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On the degree of approximation by certain linear positive operators

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1. INTRODUCTION AND SUMMARY

Let I be an arbitrary (not necessarily bounded) interval of the real axis $\mathbb{R}$. Let $UC^1(I)$ be the class of functions, defined and continuously differentiable on I and possessing a uniformly continuous derivative on I.

In this paper linear positive operators L_n $(n=1, 2, \ldots)$ of $UC^1(I)$ into itself are considered, satisfying

$$(1.1) \qquad L_n(t^i; x) = x^i \quad (x \in I;\ i = 0, 1).$$

For all $f \in UC^1(I)$ an estimation is derived for the difference $L_n(f; x) - f(x)$, $(x \in I)$, in terms of the modulus of continuity of the derivative of f defined by

$$(1.2) \qquad \omega_1(f; \delta) = \sup_{|x-y| \leqslant \delta} |f'(x) - f'(y)| \quad (\delta > 0;\ x, y \in I).$$

The main object of this paper is to show the existence of the best functions $c_n(x; \delta)$ such that for all $f \in UC^1(I)$

$$(1.3) \qquad |L_n(f; x) - f(x)| \leqslant c_n(x; \delta)\omega_1(f; \delta) \quad (\delta > 0;\ x \in I;\ n = 1, 2, \ldots)$$

and, moreover, to determine these functions. It is proved that

$$(1.4) \qquad c_n(x; \delta) = \sup_{f \in UC^1(I)} \frac{|L_n(f; x) - f(x)|}{\omega_1(f; \delta)} = L_n(\bar{f}; x),$$

where $\tilde{f}(t)$ is defined by

$$(1.5)\qquad \tilde{f}(t)=\tfrac{1}{2}|t-x|+\sum_{j=1}^{\infty}(|t-x|-j\delta)_+,$$

where $a_+=\max(a, 0)$ for $a\in\mathbb{R}$.

In view of (1.1) the linear functions will be disregarded in (1.4), as they are of no interest for the problem considered. Therefore it is always assumed that $f\in UC^1(I)$ is not linear.

The functions $\tilde{f}(t)$ are called *extremal functions.* Such functions occur for the first time in a paper by Schurer and Steutel [7], who proved (1.4) for the special case of the Bernsteinoperators. They also occur in [8], [9], [10] by the same authors, in [11] by Schurer, Sikkema, Steutel and in [3], [4] by Van der Meer. For further details the reader is referred to section 4 where some applications will be given. In section 2 of this paper three preliminary lemmas are proved. Section 3 contains the main theorem and a consequence of it in case of some special interpolation-type operators. In section 4 some applications are given and, finally, in section 5 the operators of Szász-Mirakjan are considered for which the asymptotic behaviour is determined of $c_n(x; n^{-\alpha})$ with $0\leqslant\alpha\leqslant 1$ and $x\in[0,\infty)$ if $n\to\infty$.

2. SOME PRELIMINARY LEMMAS

In this section three lemmas are derived, which are used in the proof of the main theorem and its corollary.

LEMMA 1. *Let $g(t)$ be a real function, defined and continuously differentiable on $\mathbb{R}$ with $\omega_1(g;\delta)=1$ for a fixed $\delta>0$ and let at a point $x\in\mathbb{R}$ $g(x)=0$ and $g'(x)=0$. Then there exists a real number $\lambda=\lambda(g)$ with $|\lambda|\leqslant\frac{1}{2}$ such that for all $t\in\mathbb{R}$*

$$(2.1)\qquad g(t)\leqslant\tilde{f}(t)+\lambda(t-x),$$

where $\tilde{f}(t)$ is defined by (1.5).

PROOF. The lemma is proven in two steps. In the first step we show that (2.1) holds on the interval $[x-\delta, x+\delta]$ after which we prove in the second step that (2.1) holds on $\mathbb{R}\backslash[x-\delta, x+\delta]$.

STEP 1. Consider the set A defined by

$$A=\{v\in[0, 1]|g(t)\leqslant v(t-x) \text{ on } [x, x+\delta]\}$$

and put $\lambda=-\frac{1}{2}+\inf A$. We notice that as a consequence of the properties of $g(t)$ we have $g(t)\leqslant|t-x|$ on $[x-\delta, x+\delta]$. Thus A is not empty and in case $\lambda=-\frac{1}{2}$ the function $g(t)$ satisfies (2.1) on $[x-\delta, x+\delta]$. Suppose now $\lambda>-\frac{1}{2}$. Then, as a consequence of the definition of λ, $|\lambda|\leqslant\frac{1}{2}$ and $g(t)$ satisfies (2.1) on $[x, x+\delta]$. We first assume that $g(x+\delta)=(\lambda+\frac{1}{2})(x+\delta)$. Using the properties of $g(t)$ and some elementary properties of the modulus of continuity we have for all $t\in[x-\delta, x)$

$$(2.2)\quad \begin{cases} g(t)=\int_t^x -g'(u)du\leqslant \int_t^x \{1-g'(u+\delta)\}du \\ =(x-t)-g(x+\delta)+g(t+\delta)\leqslant (x-t)-(\lambda+\frac{1}{2})\delta+(\lambda+\frac{1}{2})(t+\delta-x) \\ =\frac{1}{2}|t-x|+\lambda(t-x). \end{cases}$$

Assume now $g(x+\delta)<(\lambda+\frac{1}{2})\delta$. On account of the continuity of g', there exists a $t_1\in(x, x+\delta)$ such that $g'(t_1)=\lambda+\frac{1}{2}$ and $g(t_1)=(\lambda+\frac{1}{2})(t_1-x)$. As a direct consequence of $\omega_1(g;\ \delta)=1$ we then have $g(t)\leqslant(\lambda-\frac{1}{2})(t-x)$ on $[t_1-\delta, x)$. Finally, for all $t\in[x-\delta, t_1-\delta)$

$$(2.3)\quad \begin{cases} g(t)=\int_t^{t_1-\delta} -g'(u)du+g(t_1-\delta)\leqslant \int_t^{t_1-\delta} \{1-g'(u+\delta)\}du \\ +(\lambda-\frac{1}{2})(t_1-\delta-x)=(t_1-\delta-t)-g(t_1)+g(t+\delta) \\ +(\lambda-\frac{1}{2})(t_1-\delta-x)\leqslant(t_1-\delta-t)-(\lambda+\frac{1}{2})(t_1-x) \\ +(\lambda+\frac{1}{2})(t_1+\delta-x)+(\lambda-\frac{1}{2})(t_1-\delta-x)=(\lambda-\frac{1}{2})(t-x) \\ =\frac{1}{2}|t-x|+\lambda(t-x). \end{cases}$$

Hence (2.1) holds on $[x-\delta, x+\delta]$.

STEP 2. Suppose $t>x+\delta$. As a consequence of $\omega_1(g;\ \delta)=1$ and using some elementary properties of the modulus of continuity we have

$$(2.4)\qquad g'(u)\leqslant g'(u-j\delta)+j \text{ with } j=[(u-x)/\delta] \text{ for all } u>x+\delta,$$

[a] denoting the largest integer not exceeding a for $a\in\mathbb{R}$. Obviously $u-j\delta\in[x, x+\delta]$. Then

$$g(t)=\int_x^t g'(u)du=\sum_{j=0}^{k-1}\int_{x+j\delta}^{x+(j+1)\delta} g'(u)du+\int_{x+k\delta}^{t} g'(u)du,$$

where $k=[(t-x)/\delta]$. Hence, by means of (2.4),

$$\begin{aligned} g(t)&\leqslant \sum_{j=0}^{k-1}\int_{x+j\delta}^{x+(j+1)\delta}\{g'(u-j\delta)+j\}du+\int_{x+k\delta}^{t}\{g'(u-k\delta)+k\}du \\ &=\int_x^t[(u-x)/\delta]du+\sum_{j=0}^{k-1}\int_{x+j\delta}^{x+(j+1)\delta} g'(u-j\delta)du+\int_{x+k\delta}^{t} g'(u-k\delta)du \\ &=\int_x^t[(u-x)/\delta]du+kg(x+\delta)+g(t-k\delta) \\ &\leqslant\int_x^t[(u-x)/\delta]du+k(\lambda+\tfrac{1}{2})\delta+(\lambda+\tfrac{1}{2})(t-k\delta-x) \\ &=\sum_{j=1}^{\infty}(|t-x|-j\delta)_+ +\tfrac{1}{2}|t-x|+\lambda(t-x)=\tilde{f}(t)+\lambda(t-x), \end{aligned}$$

which proves (2.1) for $t>x+\delta$. Analogously (2.1) holds for $t<x-\delta$. This proves the lemma.

LEMMA 2. *Let $\bar{f}(t)$ be given as in* (1.5) *with $\delta>0$ and $x\in\mathbf{R}$. Then there exist real functions $g_\sigma\in UC^1(\mathbf{R})$ with $0<\sigma<\frac{1}{2}\delta$ for which $\omega_1(g_\sigma;\,\delta)=1$ and such that $g_\sigma(t)$ converges uniformly on $\mathbf{R}$ to $\bar{f}(t)$ if $\sigma\downarrow 0$.*

PROOF. For $0<\sigma<\frac{1}{2}\delta$ we construct a function $g_\sigma(t)$ on $\mathbf{R}$ possessing the following properties:

(i) $g_\sigma(x)=\frac{1}{4}\sigma\delta$ and $g_\sigma'(x)=0$.
(ii) $g_\sigma'(x+t)=\bar{f}'(x+t)$ if $(k+\sigma)\delta\leqslant t\leqslant(k+1-\sigma)\delta$ $(k=0,\ \pm1,\ \pm2,\ \ldots)$.
(iii) $g_\sigma'(x+t)$ is linear if $(k-\sigma)\delta<t<(k+\sigma)\delta$ $(k=0,\ \pm1,\ \pm2,\ \ldots)$.
(iv) $g_\sigma'(t)$ is continuous on $\mathbf{R}$.

As a consequence of (1.5) we have

$$\bar{f}'(x+t)=k+\tfrac{1}{2} \text{ if } k\delta<t\leqslant(k+1)\delta\ (k=0,\ \pm1,\ \pm2,\ \ldots). \tag{2.5}$$

From this and the properties of $g_\sigma(t)$ it follows easily that $g_\sigma\in UC^1(\mathbf{R})$ and $\omega_1(g_\sigma;\,\delta)=1$. Hence we have for $t>0$, using (i) and (ii)

$$\begin{cases} g_\sigma(x+t)-\bar{f}(x+t)=\int\limits_0^t\{g_\sigma'(x+u)-\bar{f}'(x+u)\}du+g_\sigma(x) \\ =\{\int\limits_0^{\sigma\delta}+\sum\limits_{j=1}^{k-1}\int\limits_{(j-\sigma)\delta}^{(j+\sigma)\delta}+\int\limits_{(k-\sigma)\delta}^{t}\}[g_\sigma'(x+u)-\bar{f}'(x+u)]du+\frac{1}{4}\sigma\delta, \end{cases} \tag{2.6}$$

where $k=[t/\delta]$. Using (i), (iii), (iv) and (2.5) we have

$$\int\limits_0^{\sigma\delta}\{g'(x+u)-\bar{f}'(x+u)\}du=\int\limits_0^{\sigma\delta}\{u/(2\sigma\delta)-\tfrac{1}{2}\}du=-\tfrac{1}{4}\sigma\delta \tag{2.7}$$

and for $j=1,\ 2,\ \ldots$

$$\int\limits_{(j-\sigma)\delta}^{(j+\sigma)\delta}\{g_\sigma'(x+u)-\bar{f}'(x+u)\}du=0. \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.6) it follows that

$$|g_\sigma(x+t)-\bar{f}(x+t)|=|\int\limits_{(k-\sigma)\delta}^{t}\{g_\sigma'(x+u)-\bar{f}'(x+u)\}du|$$

$$\leqslant\int\limits_{(k-\sigma)\delta}^{k\delta}\tfrac{1}{2}du=\tfrac{1}{2}\sigma\delta.$$

For $t<0$ we can prove in an analogous way the same inequality and thus we have $|g_\sigma(t)-\bar{f}(t)|\leqslant\frac{1}{2}\sigma\delta$ for all $t\in\mathbf{R}$. From this it follows that $g_\sigma(t)$ converges uniformly on $\mathbf{R}$ to $\bar{f}(t)$ if $\sigma\downarrow 0$.

The following lemma gives upper and lower bounds for $\bar{f}(t)$. For a proof the reader is referred to [7] and [8].

LEMMA 3. *For all* $t \in \mathbb{R}$, *for each fixed* $x \in \mathbb{R}$ *and each* $\delta > 0$, $\bar{f}(t)$ *satisfies the inequalities*

$$(2.9) \qquad (t-x)^2/(2\delta) \leqslant \bar{f}(t) \leqslant \delta/8 + (t-x)^2/(2\delta).$$

$$(2.10) \qquad \tfrac{1}{2}|t-x| \leqslant \bar{f}(t) \leqslant \tfrac{1}{2}|t-x| + (t-x)^2/\delta.$$

3. THE MAIN THEOREM

We consider linear positive operators L_n ($n=1, 2, \ldots$) mapping $UC^1(I)$ into itself and satisfying (1.1). Let $c_n(x;\delta)$ be defined by

$$(3.1) \qquad c_n(x;\delta) = \sup_{f \in UC^1(I)} \frac{|L_n(f;x) - f(x)|}{\omega_1(f;\delta)} \quad (\delta > 0;\ x \in I;\ n = 1, 2, \ldots).$$

As a consequence of lemmas 1 and 2 we are now able to prove the main theorem of this paper.

THEOREM 1. *For each* $x \in I$, *each* $\delta > 0$ *and all* $n = 1, 2, \ldots$

$$(3.2) \qquad c_n(x;\delta) = L_n(\bar{f};x).$$

PROOF. Let $x \in I$, $\delta > 0$ and $n = 1, 2, \ldots$ chosen arbitrarily and then kept fixed. Let $f \in UC^1(I)$ and, without loss of generality, $L_n(f;x) - f(x) \geqslant 0$.

Using the linearity of L_n and (1.1) we have

$$(3.3) \qquad L_n(f;x) - f(x) = \omega_1(f;\delta) L_n(g;x),$$

where $g(t) = \{f(t) - f(x) - (t-x)f'(x)\}/\omega_1(f;\delta)$ for all $t \in I$. We extend $g(t)$ to a function $G(t)$ defined on the whole of $\mathbb{R}$ by putting

(i) $G(t) = g(t)$ on I.
(ii) $G(t)$ is linear on $\mathbb{R} \setminus I$.
(iii) $G'(t)$ is continuous on $\mathbb{R}$.

Obviously $g(x) = G(x)$, $g'(x) = G'(x) = 0$ and $\omega_1(g;\delta) = \omega_1(G;\delta) = 1$. According to lemma 1 there exists a $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \frac{1}{2}$ such that

$$G(t) \leqslant \bar{f}(t) + \lambda(t-x) \text{ on } \mathbb{R}$$

and hence

$$(3.4) \qquad g(t) \leqslant \bar{f}(t) + \lambda(t-x) \text{ on } I.$$

From (3.3) and (3.4), using the positivity of L_n and (1.1), it then follows that

$$L_n(f;x) - f(x) \leqslant \omega_1(f;\delta) L_n(\bar{f};x).$$

Because of the fact that f was chosen arbitrarily we have

$$(3.5) \qquad c_n(x;\delta) \leqslant L_n(\bar{f};x).$$

As a consequence of lemma 2, $\bar{f}(t)$ is the uniform limit for $t \in I$ of functions in $UC^1(I)$ and thus in (3.5) equality holds. This proves the theorem.

COROLLARY. *For each $f \in UC^1(I)$, for each $x \in I$ and each $\delta > 0$*

$$(3.6)\qquad \omega_1(f;\delta)\mu_n(x)/\delta \leq |L_n(f;x)-f(x)| \leq \{\delta/8+\mu_n(x)/\delta\}\omega_1(f;\delta);$$

$$(3.7)\qquad \tfrac{1}{2}L_n(|t-x|;x)\omega_1(f;\delta) \leq |L_n(f;x)-f(x)| \leq \{\tfrac{1}{2}L_n(|t-x|;x)+2\mu_n(x)/\delta\}\omega_1(f;\delta),$$

with $\mu_n(x)=\frac{1}{2}L_n\{(t-x)^2;x\}$ and $n=1, 2, \ldots$.

PROOF. The proof of (3.6) and (3.7) is a direct consequence of (2.9) and (2.10) and the positivity of the operators L_n.

We now suppose that the operators L_n satisfy the additional condition, that for each $f \in UC^1(I)$ and each $x \in I$ we have

$$(3.8)\qquad L_n(f;x)=\sum_{k/n\in I} f(k/n)Q_{k,n}(x) \quad (n=1, 2, \ldots)$$

where $Q_{k,n}(x)$ is a positive continuous function for all k, $n=1, 2, \ldots$. For these operators the main theorem gives the following result in the special case $\delta=n^{-1}$.

THEOREM 2. *Let $\{L_n\}$ $(n=1, 2, \ldots)$ be a sequence linear positive operators, defined on $UC^1(I)$ and satisfying* (1.1) *and* (3.8). *Then for each $x \in I$ and for each $n=1, 2, \ldots$*

$$(3.9)\qquad c_n(x;n^{-1})=\tfrac{1}{2}nL_n\{(t-x)^2;x\}+\alpha(1-\alpha)/(2n),$$

with $\alpha=nx-[nx]$.

PROOF. Let $x \in I$. We consider the parabola

$$(3.10)\qquad q_x(t)=\tfrac{1}{2}n(t-x)^2+\alpha(1-\alpha)/(2n) \quad (t\in \mathbf{R},\ n=1, 2, \ldots)$$

and we shall prove that $q_x(k/n)=\tilde{f}(k/n)$ for all $k=0, \pm 1, \pm 2, \ldots$. For $k=1, 2, \ldots$ we have, using the definition of $\tilde{f}(t)$

$$(3.11)\qquad \tilde{f}(([nx]+k)/n)=\tilde{f}(x+(k-\alpha)/n)=\{(k-1)^2+(2k-1)(1-\alpha)\}/(2n) = \{k^2-2k\alpha-\alpha\}/(2n)=q_x(x+(k-\alpha)/n)=q_x(([nx]+k)/n).$$

Analogously we have for $l=0, 1, 2, \ldots$.

$$(3.12)\qquad \tilde{f}(([nx]-l)/n)=q_x(([nx]-l)/n).$$

From (3.11) and (3.12) it now follows that $\tilde{f}(k/n)=q_x(k/n)$ for all k. Hence by theorem 1, (3.8) and (3.10)

$$c_n(x;n^{-1})=L_n(\tilde{f};x)=\sum_{k/n\in I}\tilde{f}(k/n)Q_{k,n}(x) = \sum_{k/n\in I} q_x(k/n)Q_{k,n}(x)=\tfrac{1}{2}nL_n\{(t-x)^2;x\}+\alpha(1-\alpha)/(2n)$$

which proves the theorem.

4. SOME APPLICATIONS

In this section the results of the previous one will be applied to four types of well-known linear positive operators.

A. *The Bernsteinoperators* defined by $B_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f(k/n)$ with $f\in UC^1[0,1]\equiv C^1[0,1]$, $x\in[0,1]$ and $n=1, 2, \ldots$. These operators also satisfy the additional condition (3.8). As a consequence of (3.6) and (3.7) respectively we have that for each $f\in C^1[0,1]$, for each $x\in[0,1]$ and each $\delta>0$

$$x(1-x)\omega_1(f;\delta)/(2\delta n)\leqslant|B_n(f;x)-f(x)|\leqslant\{\delta/8+x(1-x)/(2\delta n)\}\omega_1(f;\delta)$$

and

$$\tfrac{1}{2}B_n(|t-x|;x)\omega_1(f;\delta)\leqslant|B_n(f;x)-f(x)|$$
$$\leqslant\{\tfrac{1}{2}B_n(|t-x|;x)+x(1-x)/(\delta n)\}\omega_1(f;\delta).$$

Theorem 2 gives that for each $x\in[0,1]$ and each $n=1, 2, \ldots$.

$$c_n(x,n^{-1})=\tfrac{1}{2}x(1-x)+\alpha(1-\alpha)/(2n)\quad(\alpha=nx-[nx]).$$

These results have also been proven, but in a different way, by Schurer and Steutel in [7], [8] and [11], the last of which in collaboration with Sikkema. In case $\delta=n^{-\frac{1}{2}}$ they have found some best constants ([7]) and furthermore they determined the asymptotic behaviour of the functions $c_n(x;n^{-\alpha})$ with $0\leqslant\alpha\leqslant1$ as $n\to\infty$ ([9]).

B. *The operators of Meyer-König and Zeller* defined by

$$M_n(f;x)=(1-x)^{n+1}\sum_{k=0}^{\infty}\binom{n+k}{k}x^kf\left(\frac{k}{n+k}\right),$$

with $f\in C^1[0,1]$, $x\in[0,1]$ and $n=1, 2, \ldots$ (see [5]).

By virtue of (3.6) and (3.7) respectively and using some estimations of $M_n\{(t-x)^2;x\}$ we have for each $x\in[0,1]$, for each $\delta>0$ and $n=2, 3, \ldots$

$$\frac{(n-1)}{2n^2\delta}x(1-x)^2\left(1+\frac{x}{n-1}\right)\leqslant M_n(\tilde{f};x)\leqslant\delta/8+\frac{1}{2n\delta}x(1-x)^2\left(1+\frac{x}{n-1}\right)$$

and

$$\tfrac{1}{2}M_n(|t-x|;x)\leqslant M_n(\tilde{f};x)\leqslant\tfrac{1}{2}M_n(|t-x|;x)+\frac{1}{n\delta}x(1-x)^2\left(1+\frac{x}{n-1}\right).$$

For a proof of these inequalities we refer to [10]. In that paper Schurer and Steutel also investigated the case $\delta=n^{-\alpha}$ $(0\leqslant\alpha\leqslant1)$.

C. *The operators of Baskakov* defined by

$$L_n(f;x)=(1+x)^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}(x/(1+x))^kf(k/n),$$

with $f\in UC^1[0,\infty)$, $x\in[0,\infty)$ and $n=1, 2, \ldots$ (see [1]).

These operators satisfy the additional condition (3.8). From (3.6) and (3.7) respectively it follows that for each $f \in UC^1[0, \infty)$, for each $x \in [0, \infty)$ and $\delta > 0$

$$x(1+x)\omega_1(f;\delta)/(2\delta n) \leqslant |L_n(f;x)-f(x)| \leqslant \{\delta/8+x(1+x)/(2\delta n)\}\omega_1(f;\delta)$$

and

$$\tfrac{1}{2}L_n(|t-x|;x)\omega_1(f;\delta) \leqslant |L_n(f;x)-f(x)| \leqslant \{\tfrac{1}{2}L_n(|t-x|;x)+x(1+x)/(\delta n)\}\omega_1(f;\delta).$$

Theorem 2 gives that for each $x \in [0, \infty)$ and each $n = 1, 2, \ldots$

$$c_n(x;n^{-1}) = \tfrac{1}{2}x(1+x)+\alpha(1-\alpha)/(2n) \quad (\alpha = nx-[nx]).$$

D. *The operators of Szász-Mirakjan* defined by

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty} f(k/n)(nx)^k/k!,$$

with $f \in UC^1[0, \infty)$, $x \in [0,\infty)$ and $n = 1, 2, \ldots$ (see [6], [13]).

These operators satisfy the additional condition (3.8). As a consequence of (3.6) and (3.7) respectively we have that for each $f \in UC^1[0, \infty)$, for each $x \in [0, \infty)$ and each $\delta > 0$

$$x\omega_1(f;\delta)/(2\delta n) \leqslant |S_n(f;x)-f(x)| \leqslant \{\delta/8+x/(2\delta n)\}\omega_1(f;\delta) \tag{4.1}$$

and

$$\tfrac{1}{2}S_n(|t-x|;x)\omega_1(f;\delta) \leqslant |S_n(f;x)-f(x)| \leqslant \{\tfrac{1}{2}S_n(|t-x|;x)+x/(\delta n)\}\omega_1(f;\delta). \tag{4.2}$$

Theorem 2 shows that for each $x \in [0, \infty)$ and each $n = 1, 2, \ldots$

$$c_n(x;n^{-1}) = \tfrac{1}{2}x+\alpha(1-\alpha)/(2n) \quad (\alpha = nx-[nx]).$$

Analogous results were proven in [3] and [4], where however different techniques for the proofs have been used.

In the next section we shall investigate for the operators S_n the functions $c_n(x;n^{-\alpha})$ with $0 \leqslant \alpha \leqslant 1$ and in particular we shall determine the asymptotic behaviour of them if $n \to \infty$.

5. THE ASYMPTOTIC BEHAVIOUR OF $c_n(x;n^{-\alpha})$

The determination of the asymptotic behaviour of $c_n(x;n^{-\alpha})$ $(0 \leqslant \alpha \leqslant 1)$ for the operators of Szász-Mirakjan can be carried out in the same way as that for the Bernsteinoperators (cf. [7] and [9]), though there are some slight complications. As in the case of the Bernsteinoperators the value $\alpha = \frac{1}{2}$ turns out to be the most complicated one. We first prove the following lemma.

LEMMA 4. *Put* $U_n(x)=\frac{1}{2}S_n(|t-x|; x)$ *with* $x\in[0,\infty)$ *and* $n=1, 2, \ldots$. *Then for each* $\eta>0$

$$\sqrt{n/x}\ U_n(x)\to(2\pi)^{-\frac{1}{2}}\quad(n\to\infty)$$

uniformly in $x\in[\eta,\infty)$.

PROOF. For $x\in[0,\infty)$ and $n=1, 2, \ldots$ we have

$$U_n(x)=\tfrac{1}{2}\{\sum_{k=0}^{l} s_{n,k}(x)(x-k/n)+\sum_{k=l+1}^{\infty} s_{n,k}(x)(k/n-x)\},$$

where $l=[nx]$ and $s_{n,k}(x)=e^{-nx}(nx)^k/k!$. Using (1.1) it follows that

$$U_n(x)=\sum_{k=0}^{l} s_{n,k}(x)(x-k/n)=xs_{n,k}(x)=n^l x^{l+1}\ e^{-nx}/l!$$

Hence

$$\sqrt{n/x}\ U_n(x)=(l+\alpha)^{l+\frac{1}{2}}\ e^{-(l+\alpha)}/l!$$

with $\alpha=nx-l$, thus α depends on n and x and $0\leqslant\alpha\leqslant 1$.

If $n\to\infty$ then $l\to\infty$ for each $x\in[\eta,\infty)$, where η is an arbitrary but fixed positive number. Using Stirling's formula we have

$$\sqrt{n/x}\ U_n(x)\to(2\pi)^{-\frac{1}{2}}(1+\alpha/l)^l\ e^{-\alpha}\ (1+\alpha/l)^{\frac{1}{2}} \tag{5.1}$$

uniformly in $x\in[\eta,\infty)$ if $n\to\infty$.

Furthermore it is obvious that uniformly in $x\in[\eta,\infty)$

$$(1+\alpha/l)^l\ e^{-\alpha}\to 1 \text{ and } (1+\alpha/l)^{\frac{1}{2}}\to 1\quad(n\to\infty). \tag{5.2}$$

The lemma then follows from (5.1) and (5.2).

In case $\alpha\neq\frac{1}{2}$ the following theorem holds.

THEOREM 3. *Let* $c_n(x; n^{-\alpha})$ *be defined as in* (1.4), *with* $L_n=S_n(n=1, 2, \ldots)$, $x\in[0,\infty)$ *and* $\alpha\in[0,1]$, $\alpha\neq\frac{1}{2}$. *If* $n\to\infty$

(i) $c_n(x; n^{-\alpha})\sim n^{-\frac{1}{2}}\{x/(2\pi)\}^{\frac{1}{2}}$ *if* $0\leqslant\alpha<\frac{1}{2}$

(ii) $c_n(x; n^{-\alpha})\sim n^{\alpha-1}x/2$ *if* $\frac{1}{2}<\alpha\leqslant 1$.

PROOF. The proof of (i) and (ii) respectively follows directly from (4.1), using lemma 4 and (4.2).

In case $\alpha=\frac{1}{2}$ the determination of the asymptotic behaviour is more complicated, as both $\bar{f}(t)$ and $\bar{f}(t)-\frac{1}{2}|t-x|$ are of degree $n^{-\frac{1}{2}}$. The central limit theorem and some applications of it are now the most obvious tools to use. In connection herewith, we first state two lemmas, for the proof of the first one we refer to [7].

LEMMA 5. *If U is a nonnegative random variable with distribution function F, then for $a \geqslant 0$*

$$E(U-a)_+ = \int_a^\infty (1-F(u))du,$$

where E denotes expectation.

LEMMA 6. *Let V_n be a Poisson random variable with expectation nx and variance $\sqrt{nx}$, where $n=1, 2, \dots$ and $x \in [\eta, \infty)$ with a fixed $\eta > 0$. If $U_n = (V_n - nx)/\sqrt{nx}$, and if $F_n(u)$ denotes the distribution function of $|U_n|$ then there exists for each $s=1, 2, \dots$ a constant C_s such that for all $u \geqslant 0$ and all $x \in [\eta, \infty)$ the inequality*

$$1 - F_n(u) \leqslant C_s u^{-2s}$$

holds.

PROOF. For $\delta > 0$, $x \in [0, \infty)$ and for each $s = 0, 1, 2, \dots$ the inequality

$$\sum_{|k/n - x| \geqslant \delta} s_{n,k}(x) \leqslant \sum_{k=0}^\infty s_{n,k}(x)((k-nx)/(n\delta))^{2s} \tag{5.3}$$

is valid. Further for

$$V_{n,r}(x) = n^{-r} S_n\{(t-x)^r; x\} \quad (r = 0, 1, 2, \dots)$$

the following recursion formula ([12])

$$V_{n,r+1}(x) = x\{V'_{n,r}(x) + nrV_{n,r-1}(x)\}, \tag{5.4}$$

holds, where

$$V_{n,0}(x) = 1, \ V_{n,1}(x) = 0. \tag{5.5}$$

From (5.4) and (5.5) it easily follows (cf. [12]) that $V_{n,r}(x)$ is a polynomial in nx with degree $[\frac{1}{2}r]$ and constant coefficients. Thus if $x \in [\eta, \infty)$ for each $s = 0, 1, 2, \dots$ there exists a constant C_s, independent of x, such that

$$V_{n,2s}(x) \leqslant C_s (nx)^s.$$

Hence with (5.3) one has

$$\sum_{|k/n - x| \geqslant \delta} s_{n,k}(x) \leqslant C_s x^s / (n^s \delta^{2s}).$$

Putting $\delta = u\sqrt{x/n}$ this shows that

$$1 - F_n(u) = \sum_{|k-nx| \geqslant u\sqrt{nx}} s_{n,k}(x) \leqslant C_s u^{-2s},$$

which proves lemma 6.

The following theorem gives the asymptotic behaviour of $c_n(x; n^{-\frac{1}{2}})$ if $n \to \infty$.

THEOREM 4. *Let $c_n(x; n^{-\frac{1}{2}})$ be defined as in* (1.4), *with $L_n = S_n$ $(n = 1, 2, \ldots)$. Then for each $x \in (0, \infty)$*

$$c_n(x; n^{-\frac{1}{2}}) \sim (x/n)^{\frac{1}{2}}\{(2\pi)^{-\frac{1}{2}} + 2 \sum_{j=1}^{\infty} \int_{j/\sqrt{x}}^{\infty} (u - j/\sqrt{x})\phi(u)du\},$$

where $\phi(u) = (2\pi)^{-\frac{1}{2}} \exp(-u^2/2)$.

PROOF. On account of theorem 1 we have for $c_n(x; n^{-\frac{1}{2}})$

$$(5.6) \qquad c_n(x; n^{-\frac{1}{2}}) = U_n(x) + \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} s_{n,k}(x)(|k/n - x| - j\sqrt{n})_+,$$

where $U_n(x)$ is defined as in lemma 4. From (5.6) it follows using lemma 5 and the notation of lemma 6 that

$$(5.7) \qquad c_n(x; n^{-\frac{1}{2}}) = U_n(x) + \sqrt{n/x} \sum_{j=1}^{\infty} \int_{j/\sqrt{x}}^{\infty} (1 - F_n(u))du.$$

Put $\Phi(u) = \int_{-\infty}^{u} \phi(v)dv$, then by the Berry-Esseen theorem (cf. [2]), $1 - F_n(u)$ converges uniformly in $u \geqslant 0$ to $2\{1 - \Phi(u)\}$ if $n \to \infty$ and also uniformly in $x \in [\eta, \infty)$ for each fixed $\eta > 0$. As a consequence of lemma 6 the integrals in the right-hand side of (5.7) converge uniformly in n, j and in $x \in [\eta, \infty)$ and also the sums in the right-hand side of (5.7) converge uniformly in n and $x \in [\eta, \infty)$. From this and lemma 4 it now follows that

$$\begin{aligned}\sqrt{n/x}\, c_n(x; n^{-\frac{1}{2}}) &\to (2\pi)^{-\frac{1}{2}} + 2 \sum_{j=1}^{\infty} \int_{j/\sqrt{x}}^{\infty} (1 - \Phi(u))du \\ &= (2\pi)^{-\frac{1}{2}} + 2 \sum_{j=1}^{\infty} \int_{j/\sqrt{x}}^{\infty} (u - j/\sqrt{x})\phi(u)du,\end{aligned}$$

for each $x \in (0, \infty)$ if $n \to \infty$.

This proves the theorem.

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